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Magnon properties of the one-dimensional quasiperiodic system

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Abstract. At low excitations, the ferromagnetic spin-wave problem of the one-dimensional quasiperiodic system is converted into the one which is analogous to the electronic and phonon problems. The ferromagnetic magnon properties are studied by the transfer-matrix technique and it is concluded that the frequency spectrum is a Cantor set. It is also shown that, at low excitations, the antiferromagnetic spin-wave problem of the one-dimensional quasiperiodic system can be converted in a similar way to the ferromagnetic one, and the antiferromagnetic magnon properties can be studied by the transfer-matrix technique.

Electronic states of the one-dimensional (1D) quasiperiodic system had been studied using the transfer-matrix technique (Kohmoto *et al* 1983, Ostlund *et al* 1983), but the interest in the electronic properties was renewed after the discovery of the icosahedral quasicrystals by Shechtman *et al* (1984). There have been two methods of investigating the electronic states of quasiperiodic systems; one is based on the continuous Schrödinger equation (Lu and Birman 1987, You 1988) and the other on the tight-binding model (or the discrete Schrödinger equation) (Kohmoto 1987 and references therein, Kohmoto *et al* 1987 and references therein).

The first method develops the 'boost-and-project' technique, i.e. it is based on the fact that quasilattices can be constructed by projecting special subsets of higher-dimensional periodic lattices onto lower-dimensional physical sub-spaces (You and Hu 1988 and references therein); the pseudo-Schrödinger equation with a periodic pseudopotential is established and the electronic behaviour may be treated as the projection of that of the Bloch pseudo-electrons in higher dimensions. Alternatively, the transfer-matrix technique may be employed in the second method to deal with the electronic properties in one dimension. In addition, the 1D phonon properties can also be studied similarly using the transfer-matrix technique (Kohmoto *et al* 1987 and references therein). In the following paper we shall make it clear that, at low excitations, the ferromagnetic spin-wave problem of the 1D quasiperiodic system can be converted into the one which is analogous to the electronic and phonon problems. Thus one can study the ferromagnetic magnon properties of the 1D quasiperiodic system by using the transfer-matrix technique. It will be also shown that, at low excitations, the anti-ferromagnetic spin-wave problem can be converted in a similar way to the ferromagnetic

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one and the antiferromagnetic magnon properties of the 1D quasiperiodic system can be studied by the transfer-matrix technique.

Consider a chain of spins $\mathbf{S}_n$, in which the magnitude of each spin is S and the spins are located successively at sites $n = 1, 2, \ldots, N\,(\rightarrow\infty)$. Two types of bonds, $J_{\rm A}$ and $J_{\rm B}$, are chosen, which are arranged quasiperiodically following the Fibonacci sequence S_∞ that is constructed recursively as $S_{l+1} = \{S_l, S_{l-1}\}$ for $l \geqslant 1$ with $S_0 = J_{\rm B}$ and $S_1 = J_{\rm A}$. For an arbitrary given N, the Heisenberg Hamiltonian is

$$H = -2\sum_{n=1}^{N} J_{n+1}\mathbf{S}_n \cdot \mathbf{S}_{n+1} \tag{1}$$

where the bond J_{n+1} is the exchange integral coupling the spins on sites n and $n+1$. In the ferromagnetic case, the bonds $J_{\rm A}$ and $J_{\rm B}$ are chosen to be positive, i.e. $J_{\rm A} > 0$ and $J_{\rm B} > 0$. The rectangular components of the spins obey the commutation rules

$$[S_i^\alpha, S_j^\beta] = \mathrm{i}\hbar\delta_{ij}\varepsilon_{\alpha\beta\gamma}S_i^\gamma \tag{2}$$

in which $\varepsilon_{\alpha\beta\gamma}$ is the completely antisymmetric Levi–Civita symbol ($\varepsilon_{yzx} = \varepsilon_{yzx} = \varepsilon_{zxy} = 1$; $\varepsilon_{yxz} = \varepsilon_{zyx} = \varepsilon_{xzy} = -1$).

According to the quantum-mechanical principle, the motion of the spin $\mathbf{S}_n$ is governed by the Heisenberg equation of motion

$$\mathrm{d}\mathbf{S}_n/\mathrm{d}t = (1/\mathrm{i}\hbar)[\mathbf{S}_n, H]. \tag{3}$$

In terms of the commutation rules (2), the equation of motion can be expressed in component form as

$$\mathrm{d}S_n^x/\mathrm{d}t = 2J_n(S_n^yS_{n-1}^z - S_n^zS_{n-1}^y) + 2J_{n+1}(S_n^yS_{n+1}^z - S_n^zS_{n+1}^y) \tag{4a}$$

$$\mathrm{d}S_n^y/\mathrm{d}t = 2J_n(S_n^zS_{n-1}^x - S_n^xS_{n-1}^z) + 2J_{n+1}(S_n^zS_{n+1}^x - S_n^xS_{n+1}^z) \tag{4b}$$

$$\mathrm{d}S_n^z/\mathrm{d}t = 2J_n(S_n^xS_{n-1}^y - S_n^yS_{n-1}^x) + 2J_{n+1}(S_n^xS_{n+1}^y - S_n^yS_{n+1}^x). \tag{4c}$$

Apparently, these three equations are non-linear. When the excitations are small enough with $S_n^x, S_n^y \ll S$, then $S_n^z \cong S$ and the product terms of S^x and S^y may be cancelled, yielding the linear equations

$$\mathrm{d}S_n^x/\mathrm{d}t = K_n(S_n^y - S_{n-1}^y) + K_{n+1}(S_n^y - S_{n+1}^y) \tag{5a}$$

$$\mathrm{d}S_n^y/\mathrm{d}t = K_n(S_{n-1}^x - S_n^x) + K_{n+1}(S_{n+1}^x - S_n^x) \tag{5b}$$

$$\mathrm{d}S_n^z/\mathrm{d}t = 0 \tag{5c}$$

with $K_n = 2J_nS$ and $K_{n+1} = 2J_{n+1}S$.

Assume that $S_n^x = u_n\exp(-\mathrm{i}\omega t)$ and $S_n^y = v_n\exp(-\mathrm{i}\omega t)$, one can write equations (5*a*) and (5*b*) as

$$-\omega S_n^x = \mathrm{i}K_n(S_{n-1}^y - S_n^y) + \mathrm{i}K_{n+1}(S_{n+1}^y - S_n^y) \tag{6a}$$

$$-\mathrm{i}\omega S_n^y = K_n(S_{n-1}^x - S_n^x) + K_{n+1}(S_{n+1}^x - S_n^x). \tag{6b}$$

Define $S^+ = S^x + iS^y$. The above two equations can be then converted into the following equation

$$-\omega S_n^+ = K_{n+1}S_{n+1}^+ + K_n S_{n-1}^+ - (K_{n+1} + K_n)S_n^+. \tag{7}$$

This equation of motion is analogous to the tight-binding model for the electronic problem with the hopping matrix elements in the Fibonacci sequence

$$t_{n+1}\psi_{n+1} + t_n\psi_{n-1} = E\psi_n \tag{8}$$

and that for the phonon problem with the masses equal and the spring constants following the Fibonacci sequence

$$-\omega^2\psi_n = K_{n+1}\psi_{n+1} + K_n\psi_{n-1} - (K_{n+1} + K_n)\psi_n. \tag{9}$$

In matrix form, equation (7) can be equally written as

$$\begin{aligned} \mathbf{S}_{n+1}^+ = \begin{bmatrix} S_{n+1}^+ \\ S_n^+ \end{bmatrix} &= \begin{bmatrix} (K_{n+1} + K_n - \omega)/K_{n+1} & -K_n/K_{n+1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S_n^+ \\ S_{n-1}^+ \end{bmatrix} \\ &= \mathbf{M}(K_{n+1}, K_n)\mathbf{S}_n^+. \end{aligned} \tag{10}$$

The spin wave at an arbitrary site is represented by

$$\mathbf{S}_{N+1}^+ = \mathbf{M}(N)\mathbf{S}_1^+ \tag{11}$$

where

$$\mathbf{M}(N) = \mathbf{M}(K_{N+1}, K_N)\mathbf{M}(K_N, K_{N-1}) \ldots \mathbf{M}(K_2, K_1). \tag{12}$$

The eigenvalues of $\mathbf{M}(N)$ are given by

$$\lambda_\pm = \tfrac{1}{2}\{\mathrm{Tr}\mathbf{M}(N) \pm [(\mathrm{Tr}\mathbf{M}(N))^2 - 4\det\mathbf{M}(N)]^{1/2}\} \tag{13}$$

where $\mathrm{Tr}\mathbf{M}(N)$ and $\det\mathbf{M}(N)$ are the trace and determinant of $\mathbf{M}(N)$, respectively. If the periodic or antiperiodic condition is applied, i.e. $\mathbf{S}_{N+1}^+ = \pm\mathbf{S}_1^+$, then $|\lambda_\pm| = 1$ and the condition for the allowed frequencies is

$$\mathrm{Tr}\mathbf{M}(N) = \pm\{1 + \det\mathbf{M}(N)\}. \tag{14}$$

Assume that N is a Fibonacci number F_l. Since $F_{l+1} = F_{l-1} + F_l$ with $F_0 = F_1 = 1$, the transfer matrix $\mathbf{M}(F_l) = \mathbf{M}_l$ satisfies the recursion relation

$$\mathbf{M}_{l+1} = \mathbf{M}_{l-1}\mathbf{M}_l \tag{15}$$

with $\mathbf{M}_1 = \mathbf{M}(K_A, K_A)$ and $\mathbf{M}_2 = \mathbf{M}(K_A, K_B)\,\mathbf{M}(K_B, K_A)$. As $\det\mathbf{M}_1 = \det\mathbf{M}_2 = 1$, then $\det\mathbf{M}_l = 1$ for $l \geq 1$ according to the recursion relation (15). The condition (14) becomes

$$x_l = \pm 1 \tag{16}$$

where $x_l = \frac{1}{2}\mathrm{Tr}\mathbf{M}_l$.

Commonly, it is required that any spin wave should not diverge, thus $|\lambda_\pm| \leq 1$ and the conditions for the bands and gaps of the magnon spectrum are respectively

$$\text{bands} \qquad |x_l| \leq 1 \tag{17}$$

$$\text{gaps} \qquad |x_l| > 1. \tag{18}$$

The frequency spectrum of the ferromagnetic magnon problem is obtained by the limit $l \to \infty$.

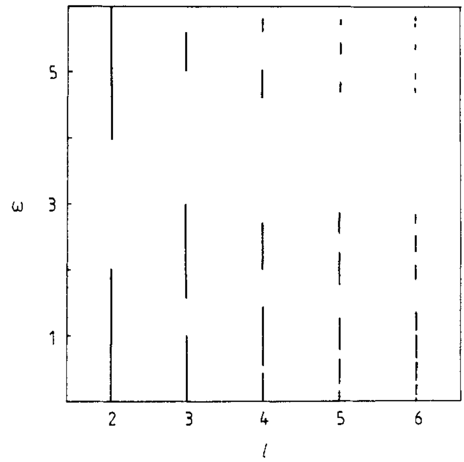

Figure 1. Band structures with $l = 2, 3, 4, 5$ and 6 for the ferromagnetic problem for $K_A = 1.0$ and $K_B = 2.0$. The spectrum of the quasiperiodic system is obtained by the limit $l \to \infty$.

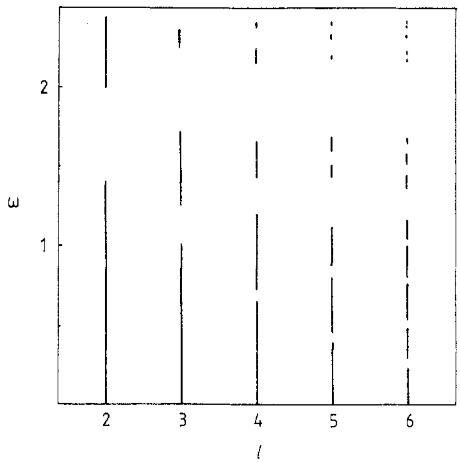

Figure 2. Band structures with $l = 2, 3, 4, 5$ and 6 for the phonon problem for $K_A = 1.0$ and $K_B = 2.0$. The spectrum of the quasiperiodic system is given by the limit $l \to \infty$.

By introducing the trace of $\mathbf{M}_l$, the matrix map (15) is reduced to the trace map (Kohmoto *et al* 1983)

$$x_{l+1} = 2x_l x_{l-1} - x_{l-2}. \tag{19}$$

The successive iteration of the trace map has a constant of motion

$$I = x_{l+1}^2 + x_l^2 + x_{l-1}^2 - 2x_{l+1}x_l x_{l-1} - 1 = (\omega^2/4)(1/K_B - 1/K_A)^2 \tag{20}$$

which is independent of the index l of the Fibonacci number. Nevertheless, the conserved quantity $I(\omega)$ depends on the frequency, i.e. the iterates evolve on different surfaces for different frequencies.

The band structures for the ferromagnetic problem are shown in figure 1 for $l = 2$, 3, 4, 5 and 6, which is similar to the spectra for the phonon problem (figure 2). The spectra for $l > 6$ can be similarly obtained and it can be shown that the spectrum of the ferromagnetic magnon problem of the 1D quasiperiodic system is a Cantor set, i.e. the spectrum has the self-similarity and the gaps in the spectrum are distributed densely. Since the ferromagnetic magnon problem is similar to the electronic problem and especially the phonon one, the conclusions derived for the electronic problem and, in particular, the phonon one are also applicable to the ferromagnetic magnon problem. Associated with the self-similarity of the spectrum, the spin wave also displays the self-similarity and at many frequencies the spin-wave functions obey the power-law scaling properties, $|S_n^+| \sim n^\beta$. At higher frequencies the bands in the spectrum are small with large gaps while at lower frequencies the bands are large with small gaps. As $l \to \infty$, the states near the upper bound of the spectrum tend to be localised, the spin-wave functions behaving locally. On the contrary, the gaps tend to vanish in the low-frequency limit $\omega \sim 0$. The bands in this region are very similar to those in the crystalline case and the states then tend to be extended with the wave-like spin-wave functions.

The antiferromagnetic Fibonacci chain of spins may be treated with suitable revision of the ferromagnetic Fibonacci chain of spins. Suppose that the spins at even quasilattice points constitute the sub-quasilattice I with the spins up (+), while the spins labelled by odd integers form the sub-quasilattice II with the spins down (−). Only the nearest-neighbour couplings are still considered and the two types of couplings, J_A and J_B, are chosen to be negative in the antiferromagnetic case.

By careful examination of equations (4*a–c*), the equation of motion at low excitations, for the sub-quasilattice I, can be written as

$$\mathrm{d}S^x_{2p}/\mathrm{d}t = K_{2p}(S^y_{2p-1} + S^y_{2p}) + K_{2p+1}(S^y_{2p} + S^y_{2p+1}) \tag{21a}$$

$$\mathrm{d}S^y_{2p}/\mathrm{d}t = -K_{2p}(S^x_{2p-1} + S^x_{2p}) - K_{2p+1}(S^x_{2p} + S^x_{2p+1}) \tag{21b}$$

$$\mathrm{d}S^z_{2p}/\mathrm{d}t = 0 \tag{21c}$$

where $K_{2p} = -2J_{2p}S = 2|J_{2p}|S$ and $K_{2p\pm1} = -2J_{2p\pm1}S = 2|J_{2p\pm1}|S$.

Assume that $S^x_n = u_n \exp(-\mathrm{i}\omega t)$ and $S^y_n = v_n \exp(-\mathrm{i}\omega t)$, as in the ferromagnetic case. Equations (21*a*) and (*b*) can be expressed as

$$\omega S^x_{2p} = \mathrm{i}K_{2p}(S^y_{2p-1} + S^y_{2p}) + \mathrm{i}K_{2p+1}(S^y_{2p} + S^y_{2p+1}) \tag{22a}$$

$$\mathrm{i}\omega S^y_{2p} = K_{2p}(S^x_{2p-1} + S^x_{2p}) + K_{2p+1}(S^x_{2p} + S^x_{2p+1}). \tag{22b}$$

These two equations can be then converted, with $S^+ = S^x + \mathrm{i}S^y$, to the following equation

$$\omega S^+_{2p} = K_{2p+1}S^+_{2p+1} + K_{2p}S^+_{2p-1} + (K_{2p+1} + K_{2p})S^+_{2p}. \tag{23}$$

Similarly, the equation of motion for the sub-quasilattice II can be converted into

$$-\omega S^+_{2p+1} = K_{2p+2}S^+_{2p+2} + K_{2p+1}S^+_{2p} + (K_{2p+2} + K_{2p+1})S^+_{2p+1}. \tag{24}$$

In matrix form, equations (23) and (24) are written respectively as

$$\begin{bmatrix} S^+_{2p+1} \\ S^+_{2p} \end{bmatrix} = \begin{bmatrix} -(K_{2p+1} + K_{2p} - \omega)/K_{2p+1} & -K_{2p}/K_{2p+1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S^+_{2p} \\ S^+_{2p-1} \end{bmatrix} \tag{25}$$

and

$$\begin{bmatrix} S^+_{2p+2} \\ S^+_{2p+1} \end{bmatrix} = \begin{bmatrix} -(K_{2p+2} + K_{2p+1} + \omega)/K_{2p+2} & -K_{2p+1}/K_{2p+2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S^+_{2p+1} \\ S^+_{2p} \end{bmatrix}. \tag{26}$$

It follows from equations (25) and (26) that the antiferromagnetic magnon properties may be also studied by the transfer-matrix technique. Just as in the crystalline case, the antiferromagnetic magnon spectrum of the 1D quasiperiodic system is different from that in the ferromagnetic case, but the feature of densely populated gaps in the frequency spectrum is still preserved. Details concerning the antiferromagnetic magnon properties of the 1D quasiperiodic system will be discussed elsewhere.

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